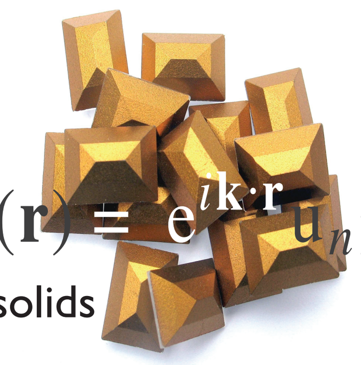
Theories, Models and Representation: Lessons from Solid State Physics

Physical theories enable us to accurately describe and predict phenomena that range from subatomic scales, to what we directly experience, to the structure and evolution of the universe. The success of these theories exposes a very strong connection between abstract mathematical structures and natural phenomena, which makes accounting for the descriptive power of mathematics a deep philosophical problem.

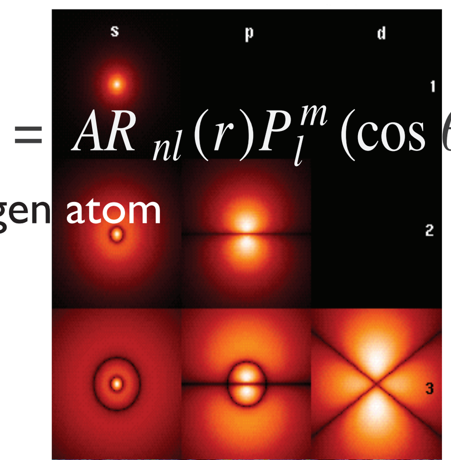
Mathematical models enable us to connect our best theories to the world and to develop an understanding of particular phenomena—they mediate between theory and phenomena. A clarification of the structure and function of models will illuminate the relationship between mathematics and the world.

Complex phenomena often require the use of multiple models to examine different features of the same phenomenon. A deeper understanding of the phenomenon is generated by examining the interconnected system of models as a whole.



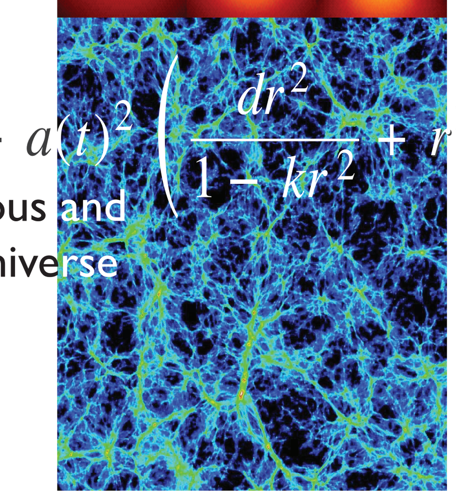
$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}u_{n\mathbf{k}}(\mathbf{r})$$

crystalline solids



$$\psi_H(r,\theta,\phi) = AR_{nl}(r)P_l^m(\cos\theta)e^{im\phi}$$

hydrogen atom



$$ds^2 = dt^2 - a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

homogeneous and isotropic universe

Understanding through Incompatible Models

Models of Crystalline Solids: Metals

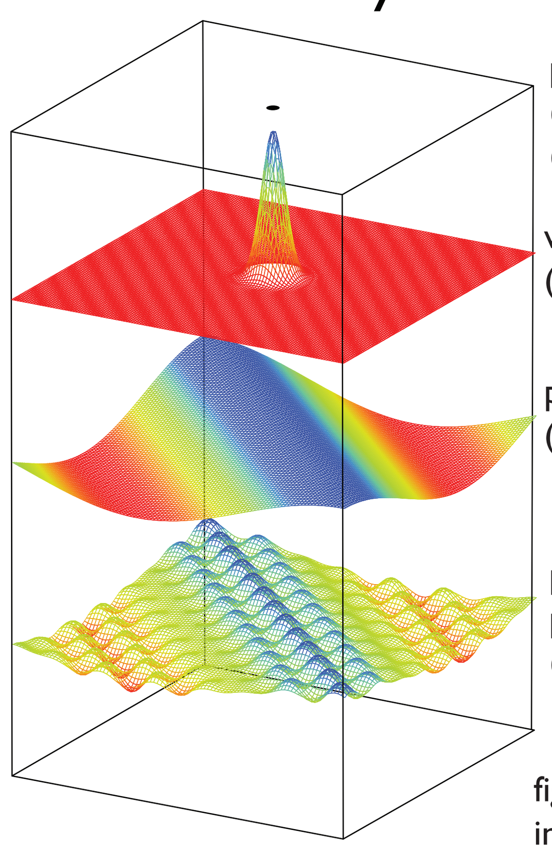


figure 1: Models of electrons in a 2D crystalline solid.

Drude Model

Electrons are point particles, which move through a static arrangement of positively charged ions (metal atoms stripped of valence electrons). The entire collection of electrons is modeled as a (classical) dilute ideal gas, using classical statistics. Electrons are assumed not to interact and, in between collisions, the electrons and ions do not interact, *i.e.* it is a *free electron* model. The Drude model had limited success experimentally.

Sommerfeld Model

The Sommerfeld model is essentially the same as the Drude model, except that quantum statistics is used. This yields a significantly better model, showing that quantum mechanics is necessary for developing a successful model of metals.

Bloch Model

Two fundamental changes: the arrangement of ions has a (Bravais) lattice structure, *i.e.* a strict periodic arrangement; and an electron-ion *interaction* is represented by a periodic potential. Electrons are now matter waves rather than point particles (see figure 1).

The Semiclassical Model

This model is built upon the Bloch model, but electrons are now treated as wavepackets of single (free) electron plane waves (see figure 1). The Semiclassical model gives genuine physical insight into what is responsible for the division of solids into metals, such as gold, insulators, such as rubber, and semiconductors, such as silicon, which are fundamental to computer processors.

The different models of solids represent the same phenomenon in incompatible ways, an example being the way an electron is modeled (see figure 1). Despite the incompatibility, the Bloch and Semiclassical models are useful for understanding different aspects of a metal (or solid). The models are also interrelated in important ways, which yield insights into the phenomena (e.g. see figure 2).

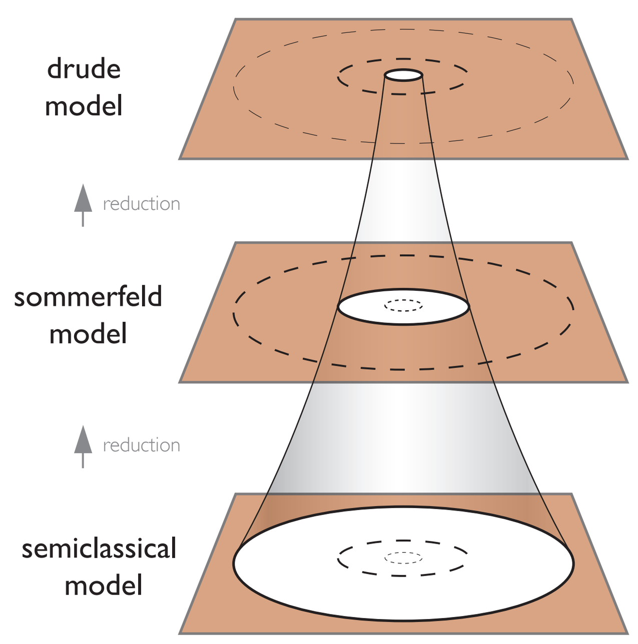
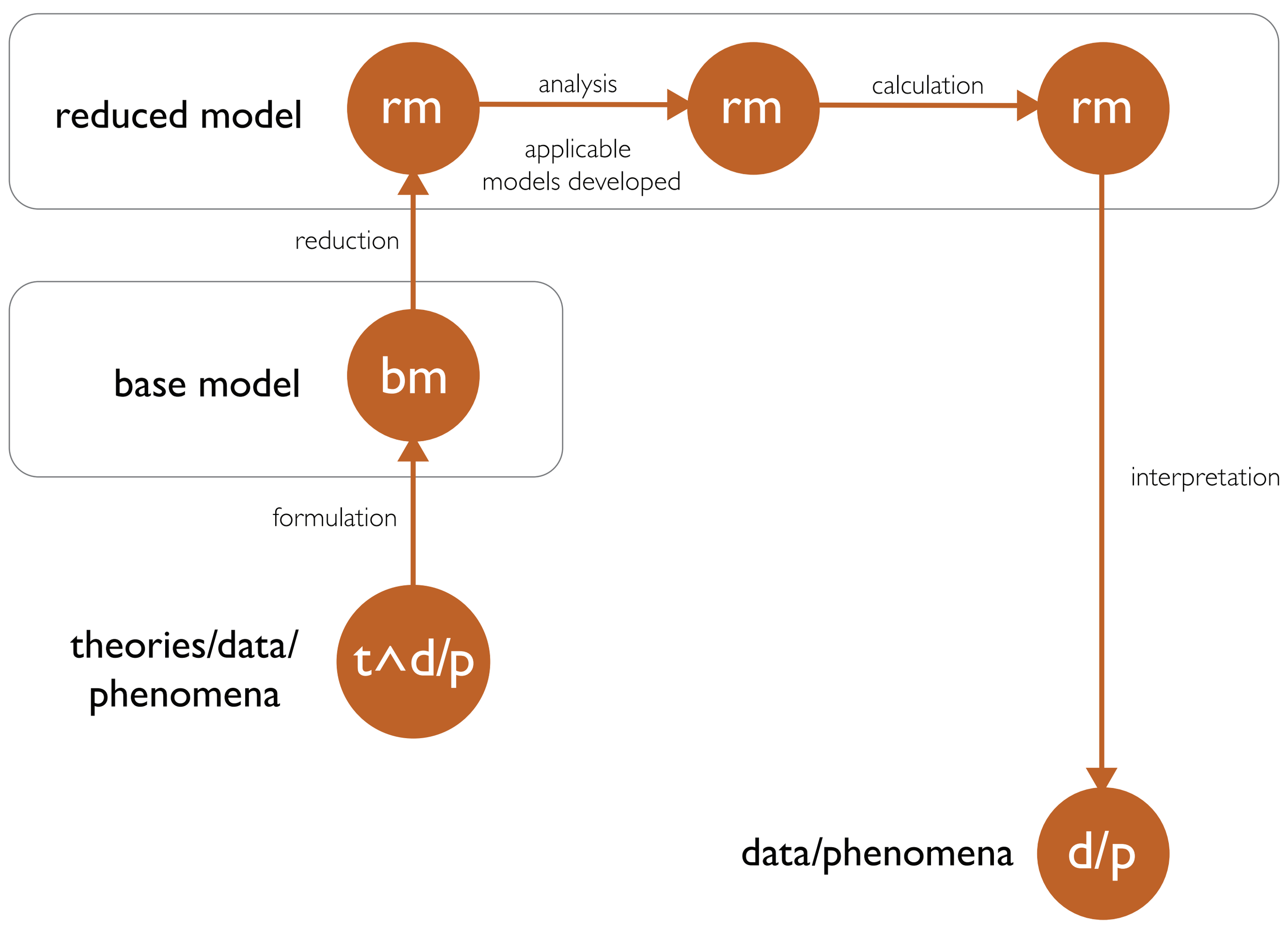


figure 2: A schematic representation of the increasing scope of progressively deeper models. The plane represents the number of phenomena successfully modeled; each deeper model accounts for significantly more phenomena.

A Model of Models: The FRACI Model

The four models motivate an account where mathematical modeling has five stages:

Formulation:	a selection of fundamental equations from theories, which is guided by the phenomenon of interest and available data.
Reduction:	this *base model* is *reduced*, i.e. equations are simplified, to yield a tractable model.
Analysis:	the equations of this *reduced model* are manipulated to study particular kinds of phenomena or model particular systems.
Calculation:	approximate results are calculated and consistency checks are performed.
Interpretation:	the results of the analysis are interpreted to describe and predict data and phenomena.



Problems with Considering the Model to Represent

A model can represent a system either abstractly or structurally. For understanding what solids *are* we are interested in structural representation.

Do the applicable models represent the actual structure of the system?

Do the models, *e.g.*, tell us what pennies are *really* like?

How Can an Electron be both One and Many?

The way that the meaning of terms, like 'electron,' shifts between models, combined with the necessity to appeal to multiple models, obscures the way in which a model is supposed to represent structure. (see figure 1).

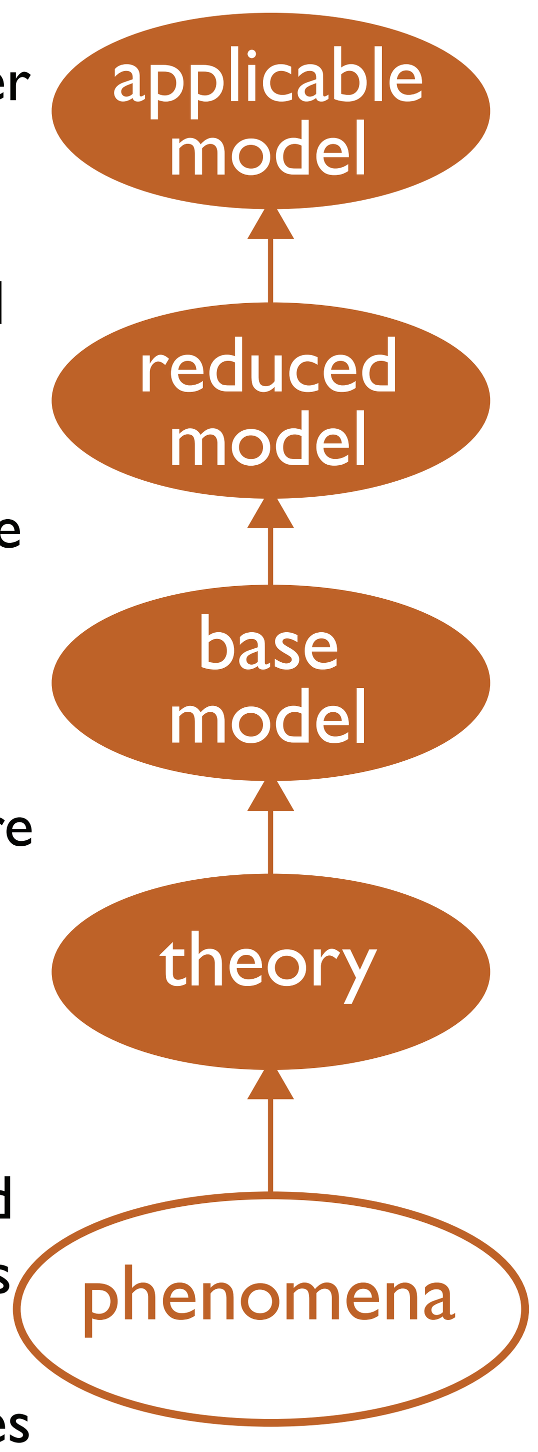
How does it make sense for an *electron* to be a wavepacket of *electrons*? This is compounded by the sheer complexity of the systems: A single penny has around one million billion billion electrons!

Despite this peculiar problem, the models yield genuine insight into the physics of the systems being investigated (*e.g.* insight into the distinction between metals and insulators (see Semiclassical model) and see figure 2). If this characterization of the situation is right, it leads to a surprising result:

Genuine physical understanding of a phenomenon is obtained from a collection of incompatible representations without a clear structural relationship to the systems modeled. This reveals just how difficult the applicability problem is.

Is there Room for Structural Representation?

There may be a way to better understand structural representation if it can be shown that the mathematical simplifications (reductions) that enable the development of applicable models preserve *structural information*. This could offer a way to explain how mathematical models represent the actual structure of physical systems, even though there is no clear structural relation between the model and the system modeled. This may also yield insight into how mathematics connects to the natural world, particularly its features that are beyond our ability to experience.



Does each arrow (transformation) preserve structural information?

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