Addition of Dedekind Reals

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Each real number is specified by a partitioning or *cut* of the rational numbers. Such a cut corresponding to the number α is designated by (A_1, A_2), where

$$A_1 \cup A_2 = \mathbb{Q} \quad \text{and } A_1 \cap A_2 = \varnothing,$$

and each member of A_1 is less than each member of A_2. The task of the addition of real numbers is to be reduced to the addition of rational numbers using the definition of reals using cuts.

To add two numbers α and β we need only to determine how to obtain the cut (C_1, C_2) corresponding to $\gamma = \alpha + \beta$ from the cuts (A_1, A_2) and (B_1, B_2).[1] The motivating idea is that if we add two rationals a_1 and b_1 from A_1 and B_1, respectively, then the result must be smaller or equal to the sum of α and β, so in C_1. Thus, if c is any rational number, it is put into the class C_1 provided that there are elements a_1 and b_1 such that $c \leq a_1 + b_1$. All other rational numbers are put into the class C_2.

This does indeed result in a cut because it is a partitioning and every number in C_1 is less than every number in C_2. To show this there are three cases to consider, but we will just consider the case where one, say α, is rational and the other, β, is irrational.

First of all for any $c_1 \in C_1$, $c_1 < \alpha + \beta$, since from $a_1 \leq \alpha$ and $b_1 < \beta$, it follows that $c_1 \leq a_1 + b_1 < \alpha + \beta$. Furthermore, suppose that there is some $c_2 \in C_2$ such that $c_2 < \alpha + \beta$. This implies that $\alpha + \beta = c_2 + \delta$, where δ is some positive real number. Let p be any positive rational number smaller than δ. Then we have $\alpha + \beta > c_2 + p$, so that

$$c_2 < (\alpha - \tfrac{1}{2}p) + (\beta - \tfrac{1}{2}p).$$

Since $\alpha - \frac{1}{2}p \in A_1$ and $\beta - \frac{1}{2}p \in B_1$, this means that $c_2 \in C_1$, which is a contradiction. Thus, every $c_2 \in C_2$, is such that $c_2 \geq \alpha + \beta$. Therefore (C_1, C_2) is indeed a cut, and is produced by the sum $\alpha + \beta$. The other two cases are shown similarly.

It follows from this that the additive structure of the rational numbers is inherited by the real numbers provided that we understand by the sum $\alpha + \beta$ of two real numbers α and β the real number γ produced be the cut (C_1, C_2).

Dedekind points out that it makes no difference to the sum in the case considered whether the number α is put in the class A_1 or the class A_2. Although this is intuitively clear, we provide a proof. In the first cut, (A_1^1, A_2^1), $\alpha \in A_1^1$. In this case C_1 is determined by those rational c_1 such that $c_1 \leq \alpha + b_1$. In the second cut, (A_1^2, A_2^2), $\alpha \in A_2^2$. In this case C_1 is

[1] A similar situation obtains for other operations on real numbers.

determined by those rational c_1 such that $c_1 \leq a_1 + b_1$, where $a_1 < \alpha$.

Suppose that $\alpha +_1 \beta \neq \alpha +_2 \beta$. In such a case it would have to be that $\alpha +_1 \beta > \alpha +_2 \beta$, since the only difference between the two cases is, respectively, whether $a_1 \leq \alpha$ or $a_1 < \alpha$. In this case, it must be that there is some $c_2^2 \in C_2^2$ such that $c_2^2 \in C_1^1$. In this case, there are $a_1^1 \in A_1^1$ and $b_1 \in B_1$ such that $c_2^2 \leq a_1^1 + b_1$. Now, either $a_1^1 < \alpha$ or $a_1^1 = \alpha$. In the former case, $c_2^2 \in C_1^2$, which is a contradiction. If, on the other hand, $a_1^1 = \alpha$, then $c_2^2 + p = \alpha + b_1$, for some positive rational number p.[2] Thus, $c_2^2 = (\alpha - \frac{1}{2}p) + (b_1 - \frac{1}{2}p)$, which again implies that $c_2^2 \in C_1^2$, a contradiction. Thus, $\alpha +_1 \beta = \alpha +_2 \beta$

References

DEDEKIND, R. 1872. *Continuity and irrational numbers, in essays on the theory of numbers*. Dover.

[2]Strictly speaking this argument will only work in case that $c_2^2 < \alpha + b_1$. In case that $c_2^2 = \alpha + b_1$, however, there will always be some $b_1' > b_1$ such that $c_2^2 < \alpha + b_1'$, since β is irrational by supposition. Thus, this case can be reduced to the one considered in the text.