The Conversion of Phenomena to Theory:
Lessons on Applicability from the Early Development of Electromagnetism

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Many considerations of the problem of the applicability of mathematics, focusing on 20th century physics, have found the successful application of abstract mathematics to physical theory in that century mysterious. A notable example is Mark Steiner who has argued that the success of the forms of argumentation used to develop quantum theories, many of which are kinds of mathematical analogy, apparently defies naturalistic explanation. Insight into the reasons for the successful application of mathematics can be gained, however, through an examination of the development of earlier theories. The consideration of 19th century physics is of particular interest since not only is this the century that saw the rise of many of the theories that would form the foundation for the development of 20th century physics, but it is in this century that physicists began to understand how to use mathematics to understand what the world is like underneath the phenomena of experience. In this paper I will examine a key period in the early development of electromagnetic theory, namely the conversion of the available knowledge of the phenomena, knowledge developed in large measure by Faraday, into a mathematical theory, primarily in the work of William Thomson and Maxwell. An examination of this episode clarifies how knowledge of phenomena is converted into a crystallized mathematical form, which provides clues as to how to account for the apparently mysterious success of mathematics as applied to 20th century physics.

1 Introduction

In recent considerations of the applicability of mathematics it is common to examine various aspects of the application of theories to natural phenomena through a consideration of the process of mathematical modeling. This has resulted in a tendency to underemphasize the role played by what underpins the entire process of modeling—scientific theories. There is a great deal that we do not understand about the relationship of theory to phenomena, especially when it comes to the peculiarities of particular theories. This paper presents the issue of how best to study the relationship between mathematical theories and natural phenomena. Since scientific theories ultimately are all developed through an attempt to develop descriptions and explanations of phenomena in experience, understanding the relationship between mathematical theories and phenomena requires first understanding how available knowledge of the phenomena is used to shape and refine the mathematical description. This may only be accomplished by examining how the theory was actually developed.

The starting point and the approach taken in an examination of the development of a theory have a significant impact on the conclusions that may be drawn about the relation of the theory to the natural phenomena it describes. Beginning with a consideration of the highly abstract theories of modern physics, and the increasingly mathematical arguments used in their development, has led some to see the success of modern physics to seem mysterious or even magical. Looking at the development of equations in quantum theory, Steiner (1998) has argued that as a result of their being developed using various kinds of mathematical, rather than physical, analogy, their significant success in applications presents a challenge to a naturalistic explanation of the applicability of mathematics. The challenge to naturalism is supposed to come as a result of mathematics being anthropocentric. I will argue in a later section that this argument is problematic for several reasons, which are evident once we better appreciate how mathematical theories of nature are developed out of experience. The successful application of the theories of modern physics cannot be properly understood through a consideration of their development alone; one must also examine the process of development of the low level theories used in their construction.

A proper understanding of the manner of the connection of physical theories, modern or otherwise, to phenomena requires at least a basic understanding of the connection of the theories to their origins in direct experience. An appreciation of the full depth of the connections of theory to experience, and hence the role played by experiential phenomena in their development, may serve as a basis for examining the connections of theory to phenomena beyond experience. This is because understanding the connections to experience will enable an elucidation of the reasoning used to develop theoretical characterizations of phenomena beyond experience, which should provide means for a clarification of the connection between the given theory and the natural phenomena to which it applies.

The difficulty in studying theory development from the point of view of understanding the relationship of physical theory to experiential phenomena is that the low level physical theories used in the majority of cases of mathematical modeling in the physical sciences have a long and complex history. Generally speaking, tracing the full connection of the mathematics of a physical theory to experiential phenomena becomes increasingly complex the more recent the theory. For example, although certain experimental anomalies were key in the development of (non-relativistic) quantum mechanics, the theory has its roots in classical mechanics, classical electrodynamics and thermal and statistical physics, which have their own complicated connections to experiential phenomena as a result of their process of development. Further difficulty arises from the fact that these theories, in turn, were developed in part on the basis of prior theory. Thus, examining how knowledge of experiential phenomena is used to develop theoretical characterizations of phenomena beyond experience becomes quite a complex affair.

This is not to suggest that understanding how theory relates to phenomena requires a full appreciation of the entire development of science. This is just to point out that modern physical theories have a deeply complex relation to the phenomena to which they apply as a result of the manner in which they, and importantly also the theories used in their construction, are developed. It is also to point out that developing a better understanding of the relationship of physical theories to natural phenomena requires a thorough look at how the low level theories in the physical sciences were developed.

The physical theories of the 19th Century are of particular interest from the point of view of understanding the relationship of physical theories to natural phenomena. This is in part because it is the theories of this century that form the basis for much of modern physics and so understanding their relationship to natural phenomena will illuminate the *physical* basis for their use of their mathematical formulation in the development of modern theories. The importance of the 19th Century is also in part a result of the fact that it was in this century that physicists learned how to develop mathematical theories of phenomena properly beyond experience. Consequently many of the theories of this century involve significant input from experiential phenomena in the service of exploring physical phenomena not present in experience. Thus, an examination of the development of the theories in this century will elucidate the manner in which mathematical characterizations of phenomena beyond experience are generated.

One 19th Century theory of interest here is electromagnetism. Electromagnetic theory describes a wide variety of experiential phenomena but the *physical* phenomenon that the theory characterizes, *i.e.* the electromagnetic field, is beyond experience. The process by which equations that characterize the physical phenomenon are developed is long and complicated and involves many different kinds of inputs and checks. A consideration of some episodes in the development of this theory will serve to clarify certain features of the process by which the equations of new physical theories are developed and the complex role played by the available knowledge of the experiential phenomena that the new theory is being developed to explain.

The focus of this paper is an examination of episodes in the early development of electromagnetic theory that illustrate features of the manner in which mathematical characterizations of physical phenomena beyond experience are developed. In the next section I consider features of the early development of electromagnetic theory, from the important role played by Michael Faraday in developing knowledge of the phenomena and in the origins of much of the theory to the work of William Thomson and James Clerk Maxwell, who created the mathematical characterization of the phenomena.

We will see that developing a mathematical theory of the phenomena beyond experience requires exploiting analogies to physical phenomena in experience and the use of dynamical models based on a particular way of conceiving of the physical constitution of the phenomenon. I will describe a process of reflective equilibrium between theory and experiential phenomena (experiment, observation, data) and the use of a form of transcendental argumentation that also occurs in the development of electromagnetism. This shows that the process of theory development is quite subtle and complex, as is the connection between theory and experiment. Given the results of the historical examination, in the subsequent section I briefly consider implications for the relationship between mathematics and natural phenomena. In particular I refute Steiner's claim that, as was apparently common in the development of quantum theories, Maxwell used purely mathematical analogies and inferred the existence of electromagnetic waves on the basis of such an analogy. I suggest that the results of this study indicate that there are lessons to be learned from a closer examination of the history from the point of view of understanding the application and applicability of mathematics in physics, including 20th Century physical theory.

2 Episodes of Theory Development in 19th Century Physics

As we shall see from the subsequent examination of the history, physicists used information available in experience in a variety of ways in order to develop new theories. There is, of course, much input coming from available knowledge of the experiential phenomena that the theory is being developed to explain. This includes the results of various experiments and empirical data as well as what I will call *empirical relations*, *i.e.* functional relations between measurable physical quantities developed from experiment or analysis of the data. I choose this name for two reasons: I want to avoid the use of the term 'law' to refer to patterns between variables determined from experiment;[1] and I want to distinguish empirically determined functional relations from relations between measurable physical quantities derived from theory, which I will call *phenomenological relations*.

As we shall see, knowledge of experiential phenomena also enters indirectly in an important way through physical analogies to more familiar phenomena. For example, analogies to the phenomena of heat flow, fluid flow and elastic media were important in the development of electromagnetic theory. In the search for equations that characterize the physical phenomenon of interest, which is not available in experience, analogies between the phenomenon of interest and more familiar physical phenomena enabled the transfer of available equations into attempts to characterize the new physical phenomenon.

In the case of electromagnetism, the physical phenomenon to be characterized by theory was considered to be the result of the motion, vibration or distortion of an imponderable aether that fills all space. Since this aether can be modeled in quite different ways, *e.g.* as a continuum characterized by a small number of parameters or as a vast collection of interacting particles, there is a distinction to be made between different ways of considering the aether analogous to that found in the context of contemporary continuum mechanics. To treat the aether as a continuous medium characterized by a small number of parameters, *e.g.* density, rigidity, compressibility, is to treat the aether *phenomenologically*. A theory or model of the aether developed on such a basis is *phenomenological*. Phenomenological theories or models treat the medium in terms of how the system behaves. To treat the aether as a vast collection of interacting particles, such that the bulk behaviour of the medium is the collective result of these interactions is to treat the aether *constitutively*, in terms of what it is made of. If the result is still to treat the overall behaviour of the medium in terms of a small number of parameters, averaging out the result of the complex details, the result is still a phenomenological model or theory. If, on the other hand, the behaviour of the system is treated fully in terms of the detailed constitution of the medium, then the result is a *constitutive* model or theory.

In the case of the *theories* to be considered in this paper, they are all phenomenological theories in the sense just detailed. In certain cases, however, constitutive models are used in the service of developing an understanding of how to think about the mathematics. Both phenomenological reasoning and constitutive reasoning are used in different ways in order to make epistemic progress. The overall process of theory development is seen to be a complex dynamic between constitutive and phenomenological reasoning, physical analogy and knowledge of empirical and phenomenological relations. We will therefore see just how involved the process of development of these new physical theories is, a process that is very far removed from the use of simple inductive generalization.

[1]The term 'law' is not appropriate in these cases since empirical equations are essentially empirical generalizations and as such do not support counterfactuals and are not capable of explaining the phenomena.

2.1 The Early Development of Electromagnetic Theory

At the turn of the 19th Century there were two distinct branches of the study of electric phenomena. There were *electrostatics*, the branch concerned with the investigation of phenomena associated with static electricity, and *electrodynamics*, the branch concerned with the study of phenomena associated with electric currents. The two branches were distinct from one another, since the two branches were associated with different experiential phenomena. Each branch also had its own tools for experimentation. Static electricity was generated with frictional generators and it could be stored using Leyden jars, an early form of capacitor. Electric currents were generated by galvanic cells, a kind of battery, which could be combined to produce a 'voltaic pile' that could produce large currents.

The early development of the theory of electricity involved competition between rival hypotheses concerning the constitution of the phenomena. I shall call such hypotheses *constitutive hypotheses*. The two constitutive hypotheses concerning the constitution of electricity arose from the study of static electricity. One hypothesis was that static electricity consisted of two imponderable fluids, vitreous and resinous, owing to the fact that rubbing glass and rubbing amber produced different kinds of static electricity. Since, as we now understand it, a positive charge develops on glass and a negative charge on amber, the phenomena associated with vitreous and resinous charge were opposite to one another, and one way of explaining this was in terms of two kinds of electricity. These two fluids were understood to be composed of large numbers of particles. The particles of the same kind of fluid all repelled one another and the particles of different fluids attracted each other. When the two fluids are combined in equal proportion they neutralize each other. This may be called the 'two-fluid' hypothesis. A competing hypothesis was that electricity consisted of a single imponderable fluid. In this case electrical phenomena were associated with an excess or deficit of this fluid over its normal distribution. These constitutive hypotheses were initially limited to electrostatics but Faraday extended them to electrodynamics by showing that all the phenomena associated with static electricity could be produced with currents and *vice versa*. This implied that both static electricity and electric currents had the same source and hence the same constitution.

At the opening of the 19th Century there was no known connection between electrical and magnetic phenomena. The first connection was made by Ørsted in 1819 in his famous experiment that showed that a current carrying wire deflects the needle of a compass. This provided the impetus for research into the connections between electricity and magnetism. Two years later Ampère made another important discovery, namely that two wires that carry current exert a force on each other. He then proceeded to conduct a number of precise experiments that determined the details of the force exerted between current carrying wires in different configurations. On the basis of these experiments Ampère developed a mathematical theory of the force laws. Ampère's work would later form the basis of the "continental approach" to the development of the mathematical theory of electromagnetism.

The next major discovery in the history of electromagnetism was Michael Faraday's discovery of magnetic induction, which will be discussed below. This was the first of many discoveries of important electromagnetic phenomena from the literally thousands of experiments he conducted, all documented in his lab notebooks. A great deal of the terminology in electromagnetism and electrochemistry was introduced by Faraday. Not only was Faraday a supremely gifted experimental physicist, but Faraday also contributed important theoretical ideas in the attempt to explain the phenomena in his investigations, ideas that would form the basis of Maxwell's development of the mathematical theory of electromagnetism. Since Faraday was not trained in mathematics, in fact he was self-educated, his ideas were not expressed mathematically. Rather, he had sophisticated intuitive explanations for electromagnetic phenomena that were important in guiding his research. That Faraday was incapable of expressing his ideas mathematically limited the initial impact of his research. It would take someone as mathematically and intuitively gifted as James Clerk Maxwell to develop Faraday's ideas into a mathematical theory.

From the point of view of examining the development of theories of natural phenomena beyond experience, electromagnetism provides a quite condensed example as a result of the fact that so much work of significance was done by so few people. A great deal of the early history of the development of the theory is covered in the work of three people: Faraday, Maxwell and William Thomson (Lord Kelvin). Faraday is responsible for developing a knowledge of the range of phenomena associated with electromagnetism and, as mentioned, his theoretical ideas were also important. The theoretical strategy initiated by Thomson and developed and refined by Maxwell then resulted in the foundational equations and notions of electromagnetic theory. Thus, in this section we will just consider snapshots of the work of these three physicists.

Michael Faraday's early work in electromagnetism was inspired by a desire to find analogies between phenomena associated with electricity moving in currents and at rest in conductors. Static electricity was known to exert a force over a distance and to possess the power of *induction*, *i.e.* of causing an opposite electrical state on nearby conductors. This led Faraday to suppose that electric currents might produce a similar effect. He expected to find that a current flowing in one circuit would induce a current in a nearby circuit. He found an induced current but to his surprise he found that it depended on the *variation* of the current in the other circuit. In further experiments Faraday found that current is also induced in a circuit when a magnet is brought near to the circuit and when the circuit is moved around in the presence of another circuit carrying a current.

Having discovered magnetic induction Faraday immediately sought an explanation for it. That magnetic powers had been represented by observing the curves produced in iron filings scattered on paper near a magnet or current carrying wire led Faraday to the notion of *lines of force*. The basic features of the notion of lines of force are highly suggested by the observation of patterns produced in iron filings. Lines of force are closed space curves, the direction of which at each point indicates the direction of the magnetic force. Given any small loop in space, the lines of force that pass through this loop determine a *tube of force* that returns into itself. Tubes of force contain information not only about the direction of the magnetic force, but also about its magnitude. This is because the product of this magnitude and the cross-section is constant along the tube. (Whittaker, 1951, 172) This enables the definition of *unit lines of force*, which are tubes of force with some fixed magnitude. Then the quantity[2] of the magnetic force can be represented by the number of unit lines of force intersecting an area perpendicular to their direction at any point. (Whittaker, 1951, 172)

[2]Whittaker (1951) uses the term 'intensity' here rather than 'quantity.' I use the term 'quantity' since in Maxwell's distinction between intensity and quantity in his geometric rendering of Faraday's ideas, which made a consistent distinction between force and flux, it is quantity that corresponds to the number of tubes of force passing through a unit area; the intensity refers to the number of equipotential surfaces intersecting a unit length of a given tube. See Darrigol (2000, 144-45).

The concept of unit lines of force enabled Faraday to explain magnetic induction in the following way. A source of magnetism, a permanent magnet or a current, generates a system of lines of magnetic force around it. A current is induced in a circuit when that circuit crosses lines of magnetic force, whatever their source. This enabled Faraday to explain the different sources of magnetic induction (varying currents, moving circuits, moving magnets) in terms of the relative motion of a circuit and the lines of magnetic force in its vicinity. His experiments did indeed show that the induction effect depended only on the relative motion. By performing experiments with identical sources of magnetism but with wires of different conductivity, Faraday showed that the quantity of current produced was directly proportional to the conductivity. This established that the relative motion of the circuit and lines of force produced an electromotive force (emf) of equal magnitude in each case, independent of the nature of the wire in the circuit; it also established that the emf produced depended only on the intersections of the lines of force and the wire. (Whittaker, 1951, 172-173) In additional experiments Faraday was able to show that the strength of the emf produced was proportional to the number of unit lines of force intersected by the wire per second.

An interesting and important theoretical question that arose in Faraday's theory was whether a rotating magnet carries its lines of magnetic force with it or not. If it does, then the lines of magnetic force are to be conceived of as somehow part of the magnet, or attached to it, which causes them to be carried with it. If, on the other hand, it does not, then although a magnet generates lines of magnetic force around it, once generated they are independent of it. In such a case a rotating magnet moves through its own lines of force. Faraday believed that this latter possibility was the case, showing that he conceived of magnetic systems as setting up a certain state in a surrounding medium that was responsible for the transmission of magnetic action.

What appears to be the source of the power of Faraday's research methodology was the combination of a flexible research method, which enabled him to easily vary experimental setups and test new ideas, and a keen physical intuition and imagination, which enabled him to conceive of a theoretical framework characterizing an underlying physical phenomenon that could be thought to be responsible for the experiential phenomena observable in his experiments. This combination enabled him to make discoveries, propose theoretical explanations and quickly test these proposals by varying the experimental setup. The ability to develop a theoretical framework, and not just an efficient and effective experimental methodology, is key here, since this is what provided a ground on which to understand the experiential phenomena he observed and to work as a basis for the prediction of new phenomena. Thus, he possessed a mastery of and ability to continuously modify the theoretical framework that guided his experimental researches. If a prediction were to fail then he could easily vary his theory to accommodate the new result. This process used by Faraday is well described as a kind of reflective equilibrium between experiment and theory, a process that enabled him to find a consistent theoretical explanation of all known experiential phenomena. This process placed very strong constraints on his theories, which accounts for their depth and robustness. It is the depth of Faraday's theories that enabled them to be used by Thomson and particularly effectively by Maxwell as the basis for the development of a mathematical theory.

Faraday also conducted a great amount of research in electrochemistry. His research into electrolysis led to further development of his research in electromagnetism. This began with his consideration of substances, such as turpentine or sulphur, that did not conduct electricity and did not decompose when placed between two electrodes with a potential difference across them. He noticed that when such substances were placed between the plates of a condenser that the charge that develops on the plates for a given potential difference depends on the nature of the substance. He called such substances *dielectrics*. Faraday's discovery that mixtures of dielectrics caused Coulomb's inverse square force law to fail to apply between the plates raised the issue of how to reconcile the ideas of electrostatic action with the behaviour of dielectrics. To address this Faraday developed a theory of the physical behaviour of dielectrics.

Building on an idea of Sir Humphry Davy's that before electrolytes are decomposed by a potential difference the liquid molecules are polarized, Faraday developed a hypothesis for the behaviour of dielectrics. He supposed that the behaviour of a dielectric is the same as that of an electrolyte up until it decomposes due to the electric stress from the applied potential. In electrolysis electric induction causes the stress on the electrolyte that ultimately leads to electrolysis and so electric induction cases the polarization of dielectrics with an applied potential. He supposed that, unlike ordinary electric induction where electric action is transmitted over a distance, induction in this case only happen through the influence of the intervening matter. (Whittaker, 1951, 186) The polarization of dielectrics was seen to be analogous to the magnetic polarization, or magnetization, developed by iron due to the action of lines of magnetic force. This led Faraday to introduce the notion of *lines of electric force* to explain the polarization of dielectrics. He defined a line of electric force to be a space curve whose tangent at each point is the direction of the electric force. (Whittaker, 1951, 186) It is worth noting here that unlike his theory of magnetic lines of force, which did not depend on any assumptions concerning the constitution of the lines of force, Faraday's theory of lines of electric force depends on a specific constitutive conception of material dielectrics.

Faraday was able to explain the failure of Coulomb's law in the case of mixed dielectrics between the plates of a condenser in terms of lines of electric force. Given a dielectric, if the charge that develops on the plates differs by a multiplicative factor ϵ from the charge that develops at the same potential with air between the plates, then ϵ is regarded as a "measure of the influence which the insulator exerts on the propagation of electrostatic action through it." (Whittaker, 1951, 184-5) Faraday called this parameter the *specific inductive capacity*. If a single dielectric is placed between the parallel plates of a condenser then the lines of electric force will be straight lines from one plate to the other. In this case Coulomb's law holds. If a mixture of dielectrics with different specific inductive capacities is placed between the plates, however, then the lines of force will tend to crowd into the regions containing the dielectric with the higher specific inductive capacity. This thus causes Coulomb's law to fail to hold between the plates.

In the case of Faraday's introduction of lines of electric force, the connection between experiential phenomena and the genesis of the idea is much less direct. Nevertheless, the need to explain the anomaly of the failure of Coulomb's law together with consistency and harmony with his theory of lines of magnetic force led him to propose a consistent explanation of the anomaly in terms of lines of electric force. Faraday developed and refined his notion of lines of electric force by studying the lines of force produced in a number of different situations. This enabled him to determine that lines of electric force behave as though they repelled each other, as if the tubes of force had an inherent tendency to dilate. (Whittaker, 1951, 187) We see here that an anomaly led to the introduction of lines of electric force but was not sufficient to determine their characteristics. Using the results of further experimentation, Faraday then learned what the implications for his theoretical ideas were. Thus, once again we see how experiential phenomena were used to develop and refine a theory of the unobservable physical phenomenon responsible for the experiential phenomena. The same kind of reflective equilibrium process described above is being used here, which we may now understand in greater detail.

Faraday's theory of lines of electric and magnetic force left open the *mechanism* by which electric and magnetic action are communicated to a distance. Faraday studiously avoided the use of constitutive hypotheses in his explanations, preferring wherever possible to explain phenomena in language that was neutral to what the constitution of both electricity and matter really was. Like his theory of dielectrics, developing ideas on the transmission of electromagnetic action was a case that required such hypotheses. A theory of the transmission of electromagnetic action would need to account for all known electromagnetic phenomena. Faraday extended his ideas of electrostatic induction in dielectrics to account for the transmission of forces giving rise to electric currents. He also conceived of the transmission of electric and magnetic action in terms of the action of intervening particles. He proposed that the transmission of electromagnetic action occurs in terms of intervening particles which assume a nonequilibrium state for some period of time as action is transmitted, which he called the *electrotonic state*. We see then that Faraday had a conception of how electromagnetic action is transmitted but it was rather vague and under-specified. A detailed dynamical model was required to develop a proper theory.

The mathematical theory of the transmission of electromagnetic action finds its beginnings in the work of William Thomson and Maxwell. There were attempts by Riemann to develop such a theory but, though his theory is consistent with the now accepted view, it was not robust enough to serve as the basis of a full theory. (Whittaker, 1951, 214) A major limitation of Riemann's theory was that it failed to take into account the properties of the intervening medium. What was required was a dynamical model of the aether to use as a basis for the development of a full theory. The work to do this began with Thomson and was fully developed by Maxwell.

Thomson's earliest work, which he began at the age of 16, made innovative use of analogies between electrostatics and heat. His use of analogy enabled him to transfer results from one area to another. Thomson's principal inspiration was Fourier's *Théorie Analytique de la Chaleur*. He noticed that electrostatic forces in air were derived from a scalar function V that satisfied the same partial differential equation, namely the Laplace equation

$$\nabla^2 V = 0,$$

as Fourier gave for the stationary temperature distribution in an infinite solid. (Darrigol, 2000, 114) He also noticed that Poisson and Fourier treated the sources (charge/heat) of the scalar field (electric potential/temperature) differently, so the methods and approaches of the two theories were different. Thomson realized that because the two theories share the same governing equation it was possible for approaches, methods and even theorems from one theory to be transferred over to the other. For instance, Coulomb's law transferred into the context of the theory of heat became a new result on the temperature distribution of point sources of heat; an intuitively obvious consequence of the physical picture of heat flow, *i.e.* Fourier's local transfer of heat, transferred into the context of electrostatics became an important theorem, namely the surface-replacement theorem. (Darrigol, 2000, 115)

Thomson's early work had a deeper consequence in that it implied that mathematically identical theories result from the assumption of action being transmitted between particles (heat) and from the assumption of action at a distance (electrostatics). This is philosophically significant since it implies that the mathematics of phenomenological theories is insensitive to radically different constitutive hypotheses. It is also deeply physically significant since it shows that forces that were usually treated as acting at a distance could be just as easily thought to be transmitted through the local action of contiguous particles, which included gravitation and electromagnetism. Indeed, Maxwell credits this work of Thomson's as the

> first [introduction] into mathematical science that idea of electrical action carried on by means of a continuous medium, which, though it had been announced by Faraday, and used by him as the guiding idea of his researches, had never been appreciated by other men of science, and was supposed by mathematicians to be inconsistent with the law of electrical action, as established by Coulomb, and built on by Poisson. (Whittaker, 1951, 241, quoting Maxwell)

Additionally, this work was important heuristically since the ability to draw analogies between theories of local action, such as the theory of heat flow, and theories of action at a distance, such as electrostatics, provided an intuitive way of thinking about how physical action usually treated as acting non-locally is transmitted to a distance, and provided a method that could be extended to develop mechanical conceptions of the transmission of electromagnetic action.

As Thomson began to appreciate the work of Faraday he recognized that Faraday's considerations of lines of electric force were similar to his considerations on heat flow. In fact, Faraday's work on lines of electric force was so detailed and precise that he had reasoned his way intuitively to the equivalents of difficult mathematical theorems. The care of Faraday's work had the consequence, as Thomson was first to recognize, that Faraday's lines of force had a direct mathematical counterpart as lines of flow of heat. (Darrigol, 2000, 117) Thomson realized that he could extend his use of analogies to bridge the gap between theories within electromagnetism that had been treated in terms of local and non-local action. For example, he showed that Faraday's work on dielectrics, which relied on action being transmitted between the particles of the material dielectric, could be treated in terms of action at a distance, resulting in a theory of dielectrics analogous to Poisson's theory of magnetic induction. (Darrigol, 2000, 117) Thomson was also able to give physical meaning to the potential function V, since he observed that it corresponded to the electric 'tension' described by Faraday, which Faraday had measured experimentally.

Maxwell was deeply influenced by the work of Thomson and Faraday. Inspired by Thomson's novel use of analogy to convert Faraday's ideas into a mathematical form, Maxwell developed Thomson's analogical method further, using it to give full mathematical expression to Faraday's ideas. Maxwell combined lines of force and equipotential surfaces, which always intersect perpendicularly, into a single representation and spaced them according to unit differences in order to represent quantitative relations geometrically. This enabled him to incorporate tubes of force with the same relationship between magnitude and cross-section as the tubes of force in Faraday's representation. It was also important that Maxwell incorporated Faraday's consistent distinction between force and flux. This occurs in Maxwell's distinction between intensity and quantity. The *quantity* of a field refers to the number of tubes of force that pass through a unit area of an equipotential surface. The *intensity* of a field refers to the number of equipotential surfaces intersected by a unit length of a tube of force.[3] This distinction was important in guiding his theory construction. (Darrigol, 2000, 145)

That Maxwell drew lines and surfaces according to fixed relations and proportions meant that Maxwell's tubes of force corresponded to Faraday's unit lines of force. This enabled Maxwell to give a precise mathematical statement of Faraday's induction law: "the induced electromotive force around a circuit is equal to the decrease of the surface integral of magnetic force across any surface bounded by the circuit." (Darrigol, 2000, 139) Maxwell used this same method to give the first expression of the so-called Ampère law, *i.e.* that "the line integral of the magnetic force on any closed curve is measured by the sum of the embraced currents." (Darrigol, 2000, 142) These two laws formed the basis of his mathematical theory of magnetism.

Using the same geometrical approach he used to mathematize Faraday's theory, Maxwell developed and extended Thomson's electrostatics/heat flow analogy. Rather than using an analogy to heat flow, Maxwell drew analogies to the flow of an incompressible fluid. He saw this as offering a more concrete analogy as a result of the fact that heat was no longer thought of as being constituted by a continuous fluid. (Darrigol, 2000, 143) He also treated a more general case, incorporating sources and sinks and considered the fluid to flow through a porous medium with variable resistance. This last assumption was to incorporate Faraday's treatment of dielectrics in terms of conducting power to lines of electric force and Faraday's explanation of phenomena associated with diamagnetic and paramagnetic materials in terms of differing conducting power to lines of magnetic force. Maxwell used this hydrodynamic analogy to give mathematical representations of different areas of electricity and magnetism. In each case the tubes of flow, the pressure and the resistance of the medium correspond to different things, which are summarized in the following table (*it is the gradient of the pressure that corresponds to the resultant magnetic force in the case of magnetism). (Darrigol, 2000, 144)

area	**tubes of flow**	**pressure**	**resistance of the medium**
electrostatics	lines of electric induction	potential	inductive capacity of a dielectric
magnetism	lines of magnetic force	resultant magnetic force*	inverse conducting power of lines of force
electric currents	lines of current	electrostatic potential or tension	electric resistance

In this same paper, published in 1846, we find special cases of some of the now familiar Maxwell equations. The vector field that is represented by Faraday's lines of force is the vector field analogous to the velocity of the fluid and so must be circuital. Maxwell found that in the electrostatic analogy without sources present the electric force $\mathbf{E}$ is represented by lines of force, but with different dielectrics present it is $\epsilon\mathbf{E}$ that is represented by lines

[3] Flux is obtained as a surface integral of the quantity and force as a line integral of the intensity.

of force, where ϵ is a scalar field that picks out the specific inductive capacity at a given location, yielding the equation

$$\nabla \cdot \mathbf{D} = 0,$$

where $\mathbf{D} = \epsilon\mathbf{E}$. In the case where there are electric sources present the analogy still works, but it becomes more complicated since sources and sinks must be introduced into the analogy and $\mathbf{D}$ is no longer circuital. Maxwell was also able to show that in the magnetic case the connection between the strength $\mathbf{J}$ of a current and the magnetic field $\mathbf{H}$ that it generates is given by the equation

$$\nabla \times \mathbf{H} = \mathbf{J}.$$

Maxwell pointed out that this equation is equivalent to the statement that "the entire magnetic intensity round the boundary of a surface measures the quantity of electric current which passes through that surface." (Whittaker, 1951, 245, quoting Maxwell) The results in this paper and their interpretations first illustrated the physical meaning of the *div* ($\nabla\cdot$) and *curl* ($\nabla\times$) operators. It was Maxwell's use of analogy and his novel geometric approach in this paper that made their physical meaning apparent.

It is important to make clear the way in which analogies were used by Thomson and Maxwell in order to give mathematical expression and rigor to Faraday's ideas. As a contrast, consider the case of the later stages of development of the wave theory of light. Here the strategy was to develop a theory of wave optics based on the equations of motion of an elastic solid aether, equations obtained by analogy to an ordinary elastic solid. The theory was supposed to give the *actual* phenomenological equations of motion of the aether in order that theoretical explanations of the experiential phenomena associated with light would be possible. In the case of the use of analogy to mathematize Faraday's theories, however, the approach is intended to yield equations *independent* of the actual constitution of the phenomenon, and independent of the physical basis of the theories to which the analogies are made. The purpose of the use of analogy is not to explain electric and magnetic phenomena but, as Maxwell himself explained, analogies were important because they provided

> a method of investigation which allows the mind at every step to lay hold of a clear physical conception, without being committed to any theory founded on the physical science from which that conception is borrowed, so that it is never drawn aside from the subject in pursuit of analytical subtleties, nor carried beyond the truth by a favorite hypothesis. (Darrigol, 2000, 147, quoting Maxwell)

Although Faraday ultimately wished to explain electromagnetic phenomena in terms of lines of force, he abstained from making constitutive hypotheses wherever possible, and his descriptions of lines of force and their behaviour is entirely phenomenological. Indeed, this is borne out by the fact that in cases where Faraday argued for a particular constitutive conception, as in the case of lines of force in dielectrics, Thomson's method of analogy showed that this constitutive conception is unnecessary.

The reason that this approach to the use of analogy was effective in this case is that Faraday's theories of lines of force were phenomenological in character. They treat the electromagnetic medium in terms of its behaviour and not in terms of its constitution. That Faraday's theories were phenomenological in this sense enabled Thomson and Maxwell to use analogies to physical theories that matched the phenomenology; the nature of the physical phenomenon on which the analogies are based is irrelevant to their being exploited to give Faraday's phenomenological theories a mathematical form. Thus, the analogies function entirely at a 'surface level' in the sense that only the mathematical form is important.

There is another feature of this kind of use of analogy that points to its being entirely of a surface level character. Maxwell's resisted flow analogies were not used to create a single mathematical description of electromagnetic phenomena. Rather, the same analogy was used to independently mathematize different branches of electromagnetism. This indeed was successful, but it emphasizes how the analogy is just functioning to guide theory development and not to provide a representation of the mechanics of the transmission of electric and magnetic action. Maxwell's work results in a patchwork of independent sub-theories. Developing a unified theory of electromagnetism required the use of analogy but in a quite different way, philosophically speaking at least, since it required a single consistent dynamical foundation.

Faraday's conception of how electric and magnetic forces were transmitted through the intervening medium was through stresses of that medium. Thomson believed that this conception could only make sense by treating the stress of the intervening medium dynamically. He made the first inroads into this by using an analogy to the stress in a strained elastic solid. The theory of an elastic solid that Thomson used was due to Stokes and was derived on a phenomenological basis in terms of elements of a continuous elastic solid. Thomson used this analogy to generate the first mathematical model of the constitution of the fields produced by electric and magnetic sources. Thomson described these fields in terms of the kinds of strain associated with different sources. From Stokes' equilibrium conditions for an incompressible elastic solid, Thomson found that the deformations produced by a point charge, a magnetic dipole and a current element are, respectively,

$$\mathbf{e} = \frac{\mathbf{r}}{r^3}, \quad \mathbf{e} = \frac{\mathbf{m}\times\mathbf{r}}{r^3}, \quad \text{and} \quad \mathbf{e} = \frac{id\mathbf{l}}{r} - \frac{1}{2}\nabla\frac{id\mathbf{l}\cdot\mathbf{r}}{r},$$

where $\mathbf{e}$ is the displacement. The first deformation was associated with the electric force of a point charge and the curls of the second and third deformations were associated with the magnetic force produced by a magnetic moment $\mathbf{m}$ and a current element $id\mathbf{l}$, respectively. (Darrigol, 2000, 127) Thus, in the magnetic case, the displacement is associated with a vector potential, of which the magnetic force is the curl. This is the vector potential $\mathbf{A}$ such that, for the magnetic induction $\mathbf{B}$,

$$\mathbf{B} = \nabla\times\mathbf{A},$$

which had been introduced independently using entirely different means by the German analyst Franz Neumann a year earlier.

We may see that this use of analogy was different in an important way from his use of the analogy between electrostatics and heat flow. There he was using the analogy to transfer results from one theory to the other, thereby using the analogy to develop both theories, and to incorporate Faraday's intuitive picture of lines of force into a rigorous mathematical theory. In this case, however, his use of analogy is much closer to the way that analogy was being used in the context of development of the wave theory of light. This is of course not for the reason that both cases involve analogies to an elastic solid, but rather because the purpose here is to provide a mathematical representation of the mechanical state of the medium that is being conceived of as transmitting electric and magnetic action. Whereas representing Faraday's lines of force as lines of flow of heat is a surface level use of analogy, independent of constitutive assumptions or hypotheses, the representation of the mechanical strain of a medium responsible for the transmission of electromagnetic action amounts, theoretically even if not metaphysically, to the introduction of a constitutive hypothesis. Even though Thomson was not committed to any physical explanation of electric and magnetic forces, (Darrigol, 2000, 127) this, as Thomson indeed recognized, amounted to an assumption concerning the constitution of the aether and how it responded to the presence of electric and magnetic sources. This work truly represents the beginning of the field concept in mathematical physics. Also, although Thomson's elastic solid analogy is statical, this work was an important step towards the development of a dynamical theory of the transmission of electromagnetic action.

In Maxwell's 1846 paper discussed above, he proposed the first analytic representation of Faraday's electrotonic state. Maxwell was looking for a way of expressing Faraday's law only in terms of forces or fields internal to the conductor. As it was stated it depended on the lines of magnetic force passing through the interior of the loop forming the circuit. Thus, Maxwell sought a way of expressing the electromotive force produced in the conductor in terms of a field representing the electrotonic state of the conductor. Maxwell recognized that he could accomplish this by using the vector potential introduced by Thomson. He called the vector potential $\mathbf{A}$ the *electrotonic intensity*. The equation

$$\nabla \times \mathbf{A} = \mathbf{B} \tag{1}$$

then entailed that "the entire electrotonic intensity round the boundary of any surface measures the number of lines of magnetic force which pass through that surface." (Whittaker, 1951, 244, quoting Maxwell) Since Maxwell was able to relate the electromotive force due to magnetic induction at a given location with $-\frac{\partial \mathbf{A}}{\partial t}$, the physical interpretation of equation (1) entails that Faraday's law can be expressed by saying that "the electromotive force on any element of a conductor is measured by the instantaneous rate of change of the electrotonic intensity on that element." (Whittaker, 1951, 244, quoting Maxwell) Although this enabled Maxwell to express Faraday's notion of electrotonic state mathematically, the treatment is still phenomenological and specific to a particular partial analogy, *i.e.* only part of magnetism is being incorporated here and electrostatics is not incorporated at all. Thus, for the reasons discussed above a different approach was necessary to develop a theory of the transmission of electromagnetic action.

Thomson had a persistent interest in developing a mechanical model that would account for all physical phenomena. Thomson's approach to this was to devise a dynamical picture of the aether and matter. Thomson was led to consider the dynamics of a fluid aether filled with molecular vortices as a result of his consideration of the Faraday effect, the capacity of a magnetic field to rotate the plane of polarization of light, which had convinced him that magnetism was a rotatory phenomenon. He argued that such a dynamical model would be the only way to account for the Faraday effect, as a result of its time asymmetry, *i.e.* that reversing the direction of propagation of light reverses the direction of rotation. (Darrigol, 2000, 132) He developed an elaborate conception of the aether, which he thought of as a dilute form of matter, that accounted for transverse vibrations identified with light, heat, electric current, Joule heating, magnetic attraction, electromagnetic induction and the Faraday effect all in terms of "a universal fluid with myriads of rotating motes which could perhaps be further reduced to permanent eddies." (Darrigol, 2000, 133, quoting Thomson) Although Thomson thought that a complete dynamical model of the aether accounting for all electromagnetic phenomena was not far from being available, he did not pursue this task himself. The task of developing such a model of the aether in terms of molecular vortices was, however, picked up and completed by Maxwell.

In 1855 Maxwell declared his intention to "discover a method of forming a mechanical conception of [Faraday's] electrotonic state adapted to general reasoning" by studying the laws of elastic solids and the motion of viscous fluids. (Whittaker, 1951, 246, quoting Maxwell) He eventually became convinced that the fact that electric current transfers electrolytes in a fixed direction indicated that electric currents are phenomena of translation and, like Thomson, regarded the Faraday effect as an indication that magnetism is a rotatory phenomenon. This, together with Faraday's ideas on the dynamical behaviour of tubes of force, *i.e.* that they tend to expand laterally and contract longitudinally, guided the development of his mechanical model. In the period of 1861-62 he succeeded in developing a mechanical model of the electromagnetic field. He was led to a model in which the state of a magnetic field is represented by the motion of a viscous fluid which rotates about lines of magnetic force.

Energetic considerations led Maxwell to identify the density of the medium with the magnetic permeability μ and the circumferential velocity of the vortices with the magnetic force $\mathbf{H}$. To enable this picture to work he introduced particles between each of the fluid vortices that acted as "idle wheels," *i.e.* they roll without sliding, enabling nearby vortices to rotate in the same direction. He considered the particles to not be otherwise constrained, which entailed that the velocity of the centre of any particle would be equal to the mean of the circumferential velocities of the vortices surrounding it. (Whittaker, 1951, 248) This yields the equation

$$\mathbf{J} = \nabla \times \mathbf{H}, \tag{2}$$

where $\mathbf{J}$ is the flux of the particles.

Maxwell was then able to show that changes in the rotatory velocity of the vortices are propagated to other locations in such a way as to be determined by the equation

$$\mu \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E}, \tag{3}$$

where $\mathbf{E}$ is the force exerted on the particles by the tangential action of the vortices and $\frac{\partial \mathbf{H}}{\partial t}$ is the rate of change of the velocity of the surrounding vortices. (Whittaker, 1951, 248) Since $\mathbf{H}$ is circuital it can be expressed as $\nabla \times \mathbf{A} = \mu \mathbf{H}$, with μ constant, and since Maxwell had shown that the induced emf is $-\frac{\partial \mathbf{A}}{\partial t}$, it follows that that

$$\nabla \times (\text{induced emf}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -\mu \frac{\partial \mathbf{H}}{\partial t} = \nabla \times \mathbf{E},$$

implying that $\mathbf{E}$ is to be identified with the induced emf. Thus, Maxwell showed that the motion of the particles constitutes an electric current, the tangential force exerted by the vortices is the induced emf and he also showed that the pressure of the particles on one another can be taken to be the tension or potential of the electricity. (Whittaker, 1951, 249)

This took care of electrodynamics. The next task was to incorporate electrostatics into the model, which involved a key innovation of Maxwell's theory. To incorporate electrostatics Maxwell assumed that the substance of the vortex cells was elastic. He assumed that when the particles are displaced from their equilibrium position they exert a tangential force on the cells, resulting in a distortion of the cells, which exert an equal and opposite compensating force on the particles. The state of the medium composing the vortex cells is assumed to represent the state of the electromagnetic field. When the force acting on the particles is removed the elastic force of the vortex cells returns the particles to their equilibrium positions. Since the change in displacement of the particles involves motion of the particles, it constitutes an electric current, though the displacement itself does not. This was a generalization of Faraday's explanation of the polarization of dielectrics due to electric induction. Where Faraday's conception of displacement relied on the presence of ponderable matter, Maxwell's displacement occurred even in the absence of ordinary matter.[4]

The displacement was connected with the accompanying electromotive force by an equation of the form

$$\mathbf{D} = \frac{1}{c_1^2}\mathbf{E}, \tag{4}$$

where c_1 is an elastic constant characteristic of the medium. (Whittaker, 1951, 251) Since the rate of change of the displacement corresponds to a current, it must be added to equation (2) yielding

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \tag{5}$$

Equations (3) and (5) are the two of the familiar Maxwell equations. That $\mathbf{H}$ is circuital, *i.e.*

$$\nabla \cdot \mathbf{H} = 0 \tag{6}$$

yields another. The final Maxwell equation, that connects the displacement with the presence of electric charge, *i.e.*

$$\nabla \cdot \mathbf{D} = \rho, \tag{7}$$

where ρ is the volume density of electric charge, was added in an 1864 paper. Thus, Maxwell had successfully used his elastico-hydrodynamical model to ground his theory of the dynamics of the electromagnetic field and derive the Maxwell equations.[5] This is an early form of his theory, but this work represents its foundation.

Thus, we see that Maxwell's use of a sophisticated constitutive model is what enabled him to give a unified mathematical treatment of electrostatics and electrodynamics. He was able to use the partial analogies that he and Thomson had developed for electrostatic and magnetic phenomena to guide the development of a single constitutive model that incorporated them all. The use of a single dynamical model was key here. Since it was a physically consistent and meaningful model, its use ensured that the quantities in the resulting set of equations would be related in a physically consistent and meaningful way. And by simultaneously incorporating the distinct sets of equations for the different branches obtained by Maxwell's hydrodynamic analogies, *viz.* by correlating the various quantities to different aspects of the model in such a way that the equations were satisfied, it ensured that the distinct sets of equations would be unified in a physically meaningful way.

[4] Whittaker (1951) points out that the term 'displacement' here refers to an actual displacement, whereas the displacement was later regarded as a change in the structure rather than the position of elements of the aether. (250)

[5] Maxwell's theory was not specified by the set of four equations that now bear his name, it required several others. Nevertheless, this shows that he had by this time successfully developed the mathematical core of a complete theory of electromagnetism.

The overall form of the argument here can be seen to be similar to that of a transcendental argument. Unlike a transcendental argument, however, where the content of experience is used to determine conditions for its possibility, here knowledge of the constraints imposed by experiential phenomena *together with certain theoretical assumptions* are used to determine a complete theoretical account. Before considering how the argument takes this sort of a transcendental form, let us review the main features of the argument.

The partial analogies to an elastic solid and an incompressible fluid developed by Thomson and Maxwell suggested that a mechanical model that incorporated both would be required to model all electromagnetic phenomena. Knowledge of the phenomena enters in in a variety of ways here. First, Faraday's theories of lines of force have deep connections with the phenomena since they were developed through the reflective equilibrium process described above. Using hydrodynamic analogies he was able to state Faraday's theories in a precise mathematical form *and* to determine features of the model that were responsible for them holding. In this way, the empirical support for the equations developed by Maxwell, support coming via Faraday's theories, implied constraints on an underlying dynamical account. Another way that knowledge of the phenomena enters is the way Maxwell used electrolysis by currents and the Faraday effect to correlate the electromotive force and the magnetic force with elements of his model. Then, with the constitutive hypothesis that lines of magnetic force could be represented by vortex cells he was able to use relationships known to hold from the equations he found from his hydrodynamic analogies in the context of particular branches of electromagnetism. This is to say, his knowledge of the equations of electrodynamics determined certain features of the constitutive model and the equations of electrostatics determined other features. By incorporating both electrodynamics and electrostatics in two different ways, one using a hydrodynamic connection and the other using an elastic connection, he was able to develop a constitutive model that incorporated both in a consistent and integrated way. The result was a theory that correlated electrostatic and electrodynamic phenomena in a way that was physically meaningful. Though this was accomplished in a manner that was not thought to represent the actual physical phenomenon, the form of the model was nevertheless highly constrained by the knowledge of the experiential phenomena associated with electromagnetism.

The way that this extended argument has a transcendental form is made clear as follows. The brilliant work of Faraday had the consequence of encoding the behaviour of the phenomena in the structure and dynamical behaviour of lines of force. He was able to do this to a significant depth as a result of the reflective equilibrium process described above. This enabled Faraday's theory of lines of force to serve as a surrogate for experiential phenomena that must be recovered by the constitutive model. After recognizing the depth and consistency of Faraday's work Maxwell was able to convert much of it (albeit in a piecemeal way) into a mathematical form. The strong empirical support for Faraday's theories meant that their mathematical expression, *i.e.* Maxwell's phenomenological equations, must be recovered by a constitutive model. Thus, the requirement of agreement with experiment, and hence the recovering of Faraday's theory, is partially converted into the requirement of recovering certain equations governing electric and magnetic phenomena. This, then, is where the transcendental-type inference comes into play. Together with the constitutive assumptions of Maxwell's mechanical model, the requirement to recover the electrodynamic and electrostatic equations determines certain features of the model, including, for example, the need to introduce the displacement current. Allowing electrodynamic and electrostatic requirements to determine distinct features of the constitutive model enabled Maxwell to devise a single consistent constitutive picture that recovered all known electromagnetic phenomena.

A striking consequence of Maxwell's model was its predictions concerning the propagation of disturbances in the electromagnetic field. In the absence of sources, equations (4) and (5) yield

$$\nabla \times \mathbf{H} = \frac{1}{c_1^2}\frac{\partial \mathbf{E}}{\partial t}.$$

Together with equation (3) this implies that

$$\nabla \times \nabla \times \mathbf{H} = \frac{1}{c_1^2}\frac{\partial}{\partial t}\left(\nabla \times \mathbf{E}\right) = -\frac{1}{c_1^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}.$$

Since $\mathbf{H}$ is circuital, it follows that

$$\nabla^2 \mathbf{H} = \frac{1}{c_1^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}. \tag{8}$$

Thus, disturbances of the electromagnetic field are propagated waves travelling with velocity c_1. He showed that the electric field vector $\mathbf{E}$ is perpendicular to $\mathbf{H}$, both in the wave front. Maxwell observed that these are precisely the equations of motion for the light vector in a medium where the velocity is c_1 and so disturbances are propagated through the electromagnetic field as waves similar to that of light.[6] Maxwell found that $c_1 = c/\sqrt{\epsilon}$, where c was a constant that occurred in the transformation from electrostatic to electrodynamic units. He compared the value for c obtained by Kohlrausch and Weber with the known value of the speed of light and found that they agreed to within 2% of each other. The coincidence of the speeds and that the disturbances obey the same equations of motion as light led Maxwell to say that "we can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena." (Whittaker, 1951, 254, quoting Maxwell) This also accounted for the different indices of refraction of different materials in terms of their having a different specific inductive capacity.

Thus, not only did Maxwell's constitutive model enable him to unify electrostatics and electrodynamics and develop a unified theory of electrodynamics and the transmission of the electromagnetic action, but his model also incorporated wave optics and predicted the existence of electromagnetic waves, of which light was a particular kind. This prediction was, of course, later well confirmed by experiment. It is also a strong and clear indication that he had found equations of motion that correctly characterize the physical phenomenon corresponding to the electromagnetic field. The phenomenological theory of the electromagnetic field he developed through an ingenious combination of physical analogy and dynamical models characterizes the unobservable physical phenomenon irrespective of its actual constitution.

3 Mathematical Access to Experientially Inaccessible Phenomena

Before we continue, let us take stock. We have seen a number of important features in the strategy of theory development in electromagnetism. Many 19th Century theories, such as

[6] Equation (8) is formally identical to the equation Poisson obtained for the circuital component of the displacement in Cauchy's equation of motion for an elastic solid used in the wave theory of light, the circuital component being the one that results in transverse vibrations of the aether.

electromagnetism, represent the period of transition from the development of theories of phenomena widely accessible in experience, such as the theories of hydrodynamics and elastic media, to theories of phenomena that are inaccessible in experience. This made use of formal, but physical, analogies between models of the dynamics of the new phenomenon and theories of more familiar phenomena. Analogies were used as an aid to the mathematization of Faraday's ideas in one way and in another way where a dynamical model of the new phenomenon was developed for the purposes of the development of a full theory by Maxwell. We also saw the importance of a kind of reflective equilibrium between theory and experiment in Faraday's experimental and theoretical work. This process was key to the development of a self-consistent theory that accounts for all experimental phenomena. Finally, we saw that a kind of transcendental reasoning is used to enable the knowledge of the experiential phenomena to determine certain features of a theory of the unobservable physical phenomenon, given certain physical assumptions concerning its constitution.

One of the important things that we find from this examination of the early development of electromagnetism is that there is a complex structure of support from experience in the strategy of theory development. The development of a theory started with a hypothesis concerning the nature of the phenomenon. The hypothesis adopted could be of a phenomenological variety, as in the case of Faraday's hypothesis that magnetic phenomena can be explained in terms of invisible lines of force, or of a constitutive variety, as in the case of Faraday's hypothesis concerning the nature of dielectrics and in the case of the one- and two-fluid hypotheses for electricity. The selection of a hypothesis was guided by drawing an analogy to some familiar or intuitive phenomenon that is suggested by experience of the experiential phenomena for which a theoretical account or explanation was sought. This is the most coarse and diffuse kind of support from experience, since it is where experience is used in abduction. More robust support from experience comes once some constitutive or phenomenological hypothesis has been adopted.

Given the adoption of a hypothesis concerning the nature of the phenomenon, attempts are made to develop this into a theory or account of the phenomenally inaccessible physical phenomenon. This is where the transcendental sort of strategy of argument comes in. The hypothesis adopted, together with general physical principles where required, determines certain features of the way that the phenomenon is to be described. Since an attempt is being made to account for certain experiential phenomena, these phenomena operate as constraints on an acceptable theoretical account. In some cases the experiential phenomena will not operate as a constraint directly, but in place of this could stand certain empirical or phenomenological relations that must be satisfied for consistency with experiment, as in other cases of theory development not considered here, or certain equations determining relationships between physical quantities, as in the case of Maxwell's dynamical model of the electromagnetic field. The theoretical assumptions together with the experiential constraints or their surrogates then determine other features of the theoretical account in order to achieve logical consistency. The way that the experiential phenomena connect, particularly if they are not used directly as a constraint on the theory, into such a transcendental type of argument can be quite complex but their role is key in determining the features of the theoretical account that are required for consistency with experiment.

This transcendental sort of argument does not necessarily result in a fully successful theory or account. The argument could require unphysical assumptions to guarantee agreement with experiment, as in the case of the boundary conditions assumed by Fresnel. The argument could also simply fail to result in a theory that is in full agreement with experiment, or it could be successful when applied to certain phenomena but fail when applied to others. In such a case changes must be made to the hypothesis concerning the nature of the phenomenon and the transcendental type of argument just described must be run again. In this way there is a back and forth between the strategies of development of the theory and the experiential phenomena that indicate that theoretical changes are required. This back and forth between theory and phenomena, or experiment, is the reflective equilibrium process I described before. This is another important part of the structure of support for a theory from experiential phenomena. In the cases considered in this paper, it is important in the development of wave optics and Faraday's theories of lines of force.

Aside from the importance of the structure of empirical support for the success of the developing theory, there is also the importance of the use of dynamical models. The development of a dynamical model of the phenomenon of interest played a central role in the final stages of developing a theory of electromagnetic phenomena in a mathematical form. One reason that dynamical models were so important was because they provide a clear physical picture of the phenomenon that is an aid to building a physical conception of a phenomenon that cannot be experienced. This aids both in the development of a mathematical theory as well as in the process of learning how to think about the mathematics. This much, however, is true of the use of formal physical analogy more generally. What made dynamical models so important for the development of a full theory is that they also provided a complete and consistent physical model that accounts for all the spatiotemporal relationships between the physical quantities included in the theory. This ensured that the full set of equations for electromagnetism generated by the model could characterize a real physical phenomenon. Furthermore, that such strong constraints were placed on the resulting set of equations as a result of the complex networks of indirect support from experiment, as we have discussed, together with the theory's striking harmony with the wave theory of optics, gave strong reasons to think that they did characterize the phenomenon. We thus see that dynamical models thus provided an essential guide to theory development.[7]

The processes of theory development I have been discussing ultimately yield a physical theory expressed in terms of mathematics that characterizes the nature of the physical phenomenon. The result is a phenomenological theory that has stood the test of time. This is because this theory applies and is applicable independently of the detailed constitution of the phenomena it describes. It makes sense that, for it to be successful at all, the result of this process is a theory that accounts for experiment irrespective of the constitutive details since we have no way of knowing what the actual constitution is. Thus, since we are unable to experience the nature of the phenomenon directly, the complex process that enables the development of the mathematical theory yields a phenomenological theory which, if consistent with experiment, together with heuristic models provides the basis for our understanding of the phenomenon. The models aid in coming to grips with what the mathematics entails about the nature of the phenomenon, but ultimately it is the mathematics that characterizes it.

[7]In the case of the development of other theories, such as wave optics, dynamical models were used to represent the actual behaviour of the unobservable phenomenon. In Maxwell's case, however, the guide from the model was entirely heuristic, *i.e.* the model itself was not thought to characterize the phenomenon. Nevertheless, the model was essential for finding a physically meaningful and consistent way of correlating the phenomena and theories of electrostatics and electrodynamics.

Thus, developing our understanding of the phenomenon occurs as a result of our developing our understanding of the mathematics.

As I mentioned in the introduction, Mark Steiner (1998) has argued that the use of mathematical analogies to develop modern physical theories, particularly quantum theories, presents a significant challenge to a naturalistic explanation for the success of the application of mathematics in the description of physical phenomena. He bases this on a claim that mathematics is anthropocentric, emphasizing the importance of the human aesthetic sense in determining what theories mathematicians develop.[8] My aim is not to address this claim fully here, but it is evident from the historical examination in this paper that the human aesthetic sense is not playing a significant role in the development of physical theories in the 19th Century. Other things that challenge the relevance of this claim are that Steiner does not carefully distinguish between pure and applied mathematics. The role of aesthetic judgement is much more dominant in the case of pure mathematics. Furthermore, the role of aesthetic judgements is not uniform within pure mathematics, being, for example, particularly dominant in the algebraic theories of the 20th Century and playing a significantly smaller role in mathematical analysis, which is the dominant branch of mathematics from the point of view of applications in applied mathematics, including physics.

The aspect of Steiner's argument I do wish to address is what Steiner claims about the use of mathematical analogies in the development of physical theories. The kind of analogies that Steiner focuses on are what he called *Pythagorean analogies*. He defines analogies that "were *then* inexpressible in any other language but that of pure mathematics." (Steiner, 1998, 3) The focus of Steiner's book is largely on the development of quantum theories but he also claims that Maxwell used Pythagorean analogies in the development of his theory. He argues that an argument Maxwell presents for the necessity of the displacement current given in Maxwell's *Treatise*, published in 1873, despite his reliance on a dynamical model in his 1862 paper where he introduced the displacement current, indicates that Maxwell's reasoning was Pythagorean. He clarifies this by saying that

> once [Maxwell] had a mathematical structure which described many different phenomena of electricity and magnetism, the mathematical structure itself, rather than anything underlying it, defined the analogy between different phenomena. The analogy, which could be adopted by other physicists (Fitzgerald, Lodge), suggested the existence of electromagnetic radiation (for which there was as yet little evidence) as an experimental phenomenon. (Steiner, 1998, 79)

This first claim in this quote seems relatively unproblematic since it is the nature of a phenomenological theory that it not rely on a particular constitutive foundation. The claim that this makes his reasoning 'Pythagorean,' however, is misleading.

Two general problems I find with Steiner's claims about Pythagorean reasoning are that he does not draw a clear line between mathematical and physical reasoning, which is related to not properly distinguishing between pure and applied mathematics, and he makes strong claims on the basis of arguments made by physicists but considered in isolation from the larger context into which those arguments fit. What Steiner calls a 'mathematical structure' or 'mathematical analogy' in the case of Maxwell's theory is a phenomenological physical theory highly constrained by experiment due to the rigorous process of development described

[8]Steiner defines naturalism as opposition to anthropocentrism.

above. That Maxwell argued that the equations he developed in his mathematization of Faraday's ideas as they stood contradicted conservation of electric charge was an indication that the mathematical structure of his theory at that point required variation in order to obtain agreement with experiment. It is the mechanical model that Maxwell constructed in his aim to unify electrostatics and electrodynamics that provided physical reasons for the introduction for the displacement current. Thus, it is not Pythagorean reasoning that led him to introduce the displacement current in the first place. That his theory, now consistent with experiment, predicted the existence of electromagnetic waves, which had little experimental support at the time, as did the displacement current which was responsible for the prediction, meant that his theory had at the time a provisional status; this does not mean that the fact that Maxwell inferred the actual existence of electromagnetic waves indicates that he believed this entirely on the basis of a 'mathematical analogy.' The fact that Maxwell's interest in the dynamical model waned is a result of the fact that the purpose of the model was to develop a consistent mathematical theory. And, again, that he could rely on the mathematical theory, independently of a particular dynamical foundation, is a result of the theory being phenomenological. In this way I think that calling his reasoning Pythagorean is misleading and probably just simply false.

Steiner does not pay proper attention to the historical details in the claims he makes about the development of electromagnetism. His consideration of Schrödinger's development of his wave equation also does not consider the original work that led him to it. The complex form of the reasoning discussed in this paper, reasoning involved in the construction of physical theories of phenomena beyond experience, indicates that additions to these physical theories were made for a variety of different reasons and that the reason was never *purely* mathematical. Even in Thomson's use of analogy to relate electrostatics and heat flow the analogy is used to translate the *physical* picture of one theory into another, leading to development of the mathematical form of the second theory. To address Steiner's arguments concerning the rampant use of Pythagorean analogies in the development of quantum theories it would be necessary to examine in detail the development of these theories and to consider his claims in detail. Nevertheless, the complexity of the physical reasoning used in the development of the theories in this paper, theories that would from part of the basis for the development of quantum theories, provides good reason to suspect that the actual reasoning used in the development of quantum theories was not *purely* Pythagorean.

On the more speculative side, the considerations of this paper provide some clues for how one might account for the success of quantum theories to the extent that their development did depend on Pythagorean analogy. What is peculiar to the development of the sort of theories considered in this paper is that, as has been pointed out, they stand at the boundary between theories of phenomena accessible in experience and theories of phenomena very far removed from experience. Furthermore, they are theories of *macroscopic* experientially inaccessible phenomena. That they deal with macroscopic phenomena allows them to be described in terms of phenomenological theories similar to the theories of phenomena in experience. Also, that they are phenomenological theories entails that they are insensitive to the details of the actual constitution of the phenomena they characterize. When such theories fail, then, it may be an indication that details of the constitution of the phenomenon have become important, as is the case with the first discoveries of quantum phenomena. In such cases, then, a theory that incorporates a characterization of the constitution of the phenomenon must be developed.

Theories of experientially inaccessible phenomena that are close to experiential phenomena, which includes (classical) electromagnetism, are largely understood in terms of their mathematical form and properties. Thus, it follows that theories that incorporate microscopic detail, being much further from experience, must rely on mathematics to an even greater degree. This does not entail that their development will occur entirely in terms of mathematical analogies, however. The mathematical form of a successful physical theory encodes the nature of the phenomenon it describes and retains connections to the phenomena to which it applies as a result. Thus, the arguments that use such theories as a basis for the development of new theories, even if the arguments are mathematical, retain a connection to the physics. For example, there is a great deal of physics encoded in the system of Maxwell equations. It will also be important to carefully assess the character of the input coming directly from experience, through new and recalcitrant experimental phenomena, since it is the requirement to account for new phenomena that is key in constraining the development of a new theory.

One other thing that could play a role in accounting for the success of mathematical analogy in physical theory is how the reflective equilibrium process and the use of forms of transcendental reasoning evolve into the context of phenomena farther removed from experience. Variation of the equations was necessary in the development of Maxwell's theory in order to obtain agreement with experiment, which was justified by a his use of a dynamical model and the sort of transcendental reasoning used. This process functioned because of the ability to correlate particular features of the model with the physical quantities appearing in the equations known to be required by experiment, and the ability to do so in such a way that the equations are satisfied by those features of the model. The connection of the equations to experiential phenomena is already quite distant here, the equations being justified by Faraday's phenomenological theories, which were developed to explain the experiential phenomena. As the phenomena being characterized become further removed from experience, experiential phenomena will play an even less direct role and it is necessary to rely on prior theory to an increasing degree as a guide and as a check of success or consistency with known phenomena. The variation of equations entirely for mathematical reasons, if that what the reasoning process is reduced to, looks very much like Pythagorean analogy, however. In this case an appreciation of the historical details will be crucial to assess the actual strategy used in theory development, to determine what the connections to experience are, and how and in what way physical reasoning is involved. As has been shown in the case of electromagnetism, the conclusions that one will draw about the manner of application and the applicability of mathematics will depend strongly on one's attention to the historical details. Thus, we may see that a deeper understanding of the relationship between mathematics and natural phenomena, and of the impressive success of the abstract reasoning used to develop quantum theories, will be available from a closer examination of the development of the low level theories used in contemporary physics.

4 Conclusion

A detailed examination of episodes in the early development of electromagnetism shows that the process of reasoning used to develop physical theories is intricate and complex. It is vastly more complex than the simplistic model of simple inductive generalization leading to the development of general laws. Also it is seen that the structure of the support coming from experiential phenomena is quite complex and may occur through a chain of theoretical connections. From the point of view of understanding the actual structure of argumentation used in physical theory development, it is evident that much more attention must be paid to the details of the actual historical development of physical theories. The results of this examination show that there are patterns in theory development to be described. Since it is not clear from this examination alone what forms of reasoning occur more generally it evidently will require a quite broad examination of the history in order to achieve a more full understanding of the actual reasoning used in theory development.

From the point of view of accounting for the successful application of mathematics in its application to natural phenomena, the consideration of the history and the considerations of the previous section indicate that the justification for the development of new theories and the introduction of new theoretical ideas or concepts it quite subtle and complex. It is not possible to make strong claims about the nature of the application or the applicability of mathematics in the context of a physical theory without due consideration of the manner in which the theory was actually developed. This aspect surely generalizes to other cases of theory development, which suggests that a better understanding of the applicability of mathematics in the context of 20th Century physical theories will come from a much more detailed examination of the reasoning involved in the development of these theories.

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